Feb. 2010 CHINESE JOURNAL OF ENGINEERING MATHEMATICS Vol. 27 No. 1

Article ID:1005-3085(2010)01-0161-07

On the Solution of a Sylvester Equation $AV + BW = EVF$ Appearing in Singular Control Theory*

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Abstract: This paper deals with the solution of the matrix equation $AV+BW=EVF$ associated to a linear singular systems and subject to some rank and regional pole-placement constraints. Under the Γ-strongly controllability of the singular system, a sequence of coordinate transformations is proposed such that the considered problem can be solved through a Sylvester equation associated to a controllable reduced-order normal system. The solution of the Sylvester equation can also be found by the regional pole-placement in LMI regions of the complex plane. The results provide great convenience to the computation and analysis of the solutions to this class of equations.

Keywords: singular systems; Sylvester equation; regional pole-placement

Classification: AMS(2000) 65M06; 65M15 **CLC number:** O241 **Document code:** A

1 Introduction

Consider the singular system $(E,\ A,\ B)$ described by the following equation

$$\begin{cases} E\dot{x} = Ax + Bu, \\ y = Cx. \end{cases} \tag{1}$$

Here $x \in R^n,\ u \in R^n,\ E,\ A \in R^{n\times n},\ B \in R^{n\times p},\ C \in R^{m\times n},\ \text{rank}E = q,\ \text{rank}B = p < q$ and $\text{rank}C = m$. As in singular systems (1), many problems, such as observer design[1,2], and the problem of eigenstructure assignment[3-6], are closely related with the matrix equation of the following form

$$AV + BW = EVF. \tag{2}$$

Here A, B, E are given matrices, while $F \in R^{(q-p)\times(q-p)}$; $V \in C^{n\times(q-p)}$ and $W \in C^{p\times(q-p)}$ are to be determined; F is in the Jordan form with arbitrary given eigenvalues. In [7], Eugenio and Vilemar have obtained an explicit solutions for (2). Duan[3-4] gave a complete analytical restriction free parametric solution of (2), and then presented a new solution of (2). These existing solutions are applicable in some problems like eigenstructure assignment. However, many strict constraint conditions are imposed and large computational load are needed to

Received: 22 Oct 2008. **Biography**: Gong Wenzhen (Born in 1957), Male, Associate Professor.
Accepted: 25 May 2009. Research field: differential equations.
*__Foundation item:__ The Natural Science Foundation of China (60564001); the Program for New Century Excellent Talents in University China (CET-06-0756).

obtain these solutions of (2) in the above literature. In this note, a new and simple approach to solve matrix equation (2) is developed. As a by-product, we show that this matrix equation (2) corresponds to the solution of some known Lyapunov like equations that allows to obtain regional pole-placement in LMI-type regions of the complex plane. The structure of this paper is as follows. The next section describes some background material and specific mathematical tools needed for the algorithm development. Section 3 presents the proposed solution of matrix equation (2). A necessary and sufficient condition is first derived for the existence of the solution for (2). Then an algorithm is presented to compute the solution of (2) by the eigenstructure assignment. In section 4, a numerical example is presented. In section 5, a conclusion ends the paper.

2 Preliminaries

Assume that matrix $E \in R^{n\times n}$ is singular and rank$E = q < n$, then there exist two nonsingular matrices P, Q with appropriate dimensions such that

$$E_C = PEQ = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad A_C = PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_C = PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Let Γ be a region in the open left half complex plane, $\Gamma \subseteq C^-$, symmetric with respect to the real axis. Let $L \in R^{n\times(n-q)}$ is any full column rank matrix satisfying $EL = 0$. Let

$$Q^{-1}L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

according to this partition, $EL = 0$ becomes $E_C \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = 0$, which leads to $L_1 = 0$ and $L_2 \in R^{(n-q)\times(n-q)}$ is an arbitrary nonsingular matrix.

Definition 1 System (1) is said to be Γ-strongly controllable if and only if the following conditions are satisfied

$$\text{rank}(\lambda E - A \quad B) = n, \quad \forall\, \lambda \in C, \quad \lambda \notin \Gamma, \tag{3}$$

$$\text{rank}(E \quad AL \quad B) = n. \tag{4}$$

3 Main results

3.1 The solution of matrix equation

Let

$$\widetilde{B}_1 = (A_{12} \quad B_1), \quad \widetilde{B}_2 = (A_{22} \quad B_2),$$

$$\widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix} \in R^{n\times(n-q+p)}, \quad \overline{V} = PEV = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}, \quad T = (0 \quad I_{n-q}),$$

we obtain the following Theorem.

Theorem 1 Assume that the matrix $\widetilde{B}$ is of full column rank, that is, rank$\widetilde{B} = n - q + p$, the necessary and sufficient condition for the existence of the solution to the matrix equation (2) is that system (1) is Γ-strongly controllable.

The proof of theorem1 needs the following Lemma 1.

Lemma 1 The system (1) is Γ-strongly controllable if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda I - A_C & \widetilde{B} \\ T & 0 \end{bmatrix} = 2n - q, \quad \forall\, \lambda \in C, \quad \lambda \notin \Gamma \quad \text{and} \quad (A_{22} \ \ B_2) = n - m.$$

Proof To establish the sufficient and necessary conditions, we consider the following equations

$$\operatorname{rank}(\lambda E - A \ \ B) = \operatorname{rank}(\lambda E_C - A_C \ \ B_0) = \operatorname{rank}\begin{bmatrix} \lambda I - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} \tag{5}$$

and

$$\operatorname{rank}(E \ \ AL \ \ B) = \operatorname{rank}(PEQ \ \ PAQQ^{-1}L \ \ PB) = \operatorname{rank}\begin{bmatrix} I_q & A_{12}L_2 & B_1 \\ 0 & A_{22}L_2 & B_2 \end{bmatrix}. \tag{6}$$

Definition 1, (5) and (6) imply Lemma 1.

Proof of Theorem 1: The proof is constructive.

Sufficiency If system (1) is Γ-strongly controllable. Under the assumption that the matrix $\widetilde{B}$ is of full column rank, we can choose a matrix $R \in R^{n\times(q-p)}$ such that rank$(R \ \ \widetilde{B}) = n$. Let

$$(R \ \ \widetilde{B})^{-1} = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}.$$

From Lemma 1, we have that

$$\operatorname{rank}(A_{22} \ \ B_2) = \operatorname{rank}\widetilde{B}_2 = \operatorname{rank}T\widetilde{B} = \operatorname{rank}(TR \ \ T\widetilde{B}) = \operatorname{rank}T = n - q. \tag{7}$$

Let

$$X = M_1 A_C R - M_1 A_C \widetilde{B}(T\widetilde{B})^+ TR, \tag{8}$$

$$Z = M_1 = A_C \widetilde{B}(I - (T\widetilde{B})^+(T\widetilde{B})), \tag{9}$$

$$U_1 = \begin{bmatrix} R & \widetilde{B} & 0 \\ 0 & 0 & I \end{bmatrix}, \quad U_2 = \begin{bmatrix} I & 0 & 0 \\ (T\widetilde{B})^+ TR & I - (T\widetilde{B})^+(T\widetilde{B}) & (T\widetilde{B})^+ \end{bmatrix}, \quad U_3 = \begin{bmatrix} M_1 & 0 \\ N_1 & 0 \\ 0 & I \end{bmatrix}, \tag{10}$$

where A^+ denotes any generalized inverse of matrix, satisfying $AA^+A = A$. Consider the following equations

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} \lambda I - A_C & \widetilde{B} \\ T & 0 \end{bmatrix} &= n - q + \operatorname{rank}\begin{bmatrix} \lambda I - M_1 A_C R & M_1 A_C \widetilde{B} \\ TR & T\widetilde{B} \end{bmatrix} \\ &= 2n - 2q + p + \operatorname{rank}(\lambda I - X \ \ Z).\end{aligned}$$

From Lemma 1 and (8), we get

$$\mathrm{rank}(\lambda I - X \quad Z) = q - p, \quad \forall\, \lambda \in C, \quad \lambda \notin \Gamma. \tag{11}$$

From (9), for any given $F \in R^{(q-p)\times(q-p)}$, $\sigma(F) \subset \Gamma$, we can choose the appropriate matrix Y, such that

$$F = X + ZY. \tag{12}$$

Taking

$$K = (T\widetilde{B})^{+}TR - \big(I - (T\widetilde{B})^{+}(T\widetilde{B})\big)Y, \tag{13}$$

then we have $T\widetilde{B}K = TR$, or

$$T(R - \widetilde{B}K) = 0. \tag{14}$$

(13) is written as

$$F = M_1A_CR - M_1A_C\widetilde{B}K. \tag{15}$$

Let

$$K_1 = N_1A_CR - N_1A_C\widetilde{B}K, \tag{16}$$

By combining (13), (16) and (17), we get

$$RF + \widetilde{B}K_1 + A_C\widetilde{B}K = A_CR, \tag{17}$$

or

$$A_C(R - \widetilde{B}K) - (R - \widetilde{B}K)F = \widetilde{B}J, \tag{18}$$

where

$$J = -\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = K_1 + KF, \quad J_1 \in R^{(n-q)\times(q-p)}, \quad J_2 \in R^{p\times(q-p)}, \tag{19}$$

$$(R - \widetilde{B}K) = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \quad R_1 \in R^{q-(n-q)}. \tag{20}$$

Substituting (17) and (18) into (16) yields

$$\begin{bmatrix} A_{11}R_1 \\ A_{21}R_1 \end{bmatrix} + \begin{bmatrix} A_{12}J_1 + B_1J_2 \\ A_{22}J_1 + B_2J_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} F. \tag{21}$$

Let

$$V = Q\begin{bmatrix} R_1 \\ J_1 \end{bmatrix}, \quad W = J_2.$$

From (20), we get that

$$PAQQ^{-1}V + PBW = PEQQ^{-1}VF, \tag{22}$$

which prove the sufficiency of Theorem 1.

Necessity From the proof of sufficiency, we get that $F = X + ZY \subset \Gamma$ if and only if (9) holds, and the fact (9) holds if and only if

$$\mathrm{rank}\begin{bmatrix} \lambda I - A_C & \widetilde{B} \\ T & 0 \end{bmatrix} = 2n - q, \quad \forall\, \lambda \in C, \quad \lambda \notin \Gamma \quad \text{and} \quad (A_{22} \;\; B_2) = n - q,$$

and combining Lemma 1, we easily obtain the proof of necessity.

3.2 Algorithm

Let the conditions of Theorem 1 be satisfied, the algorithm for the solution of the matrix equation (2) can be described as follows.

Step 1 Compute matrices P, Q such that $PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, then compute

$$A_C = PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} B_0 = PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad Q^{-1}V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Step 2 Choose a matrix R such that $\mathrm{rank}(R \;\; \widetilde{B}) = n$, compute $(R \;\; \widetilde{B})^{-1}$ and let

$$(R \;\; \widetilde{B})^{-1} = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}.$$

Compute X and Z from (8) and (9), respectively. For given $F \subset \Gamma$, using the pole placement method in the context state space systems to compute the gain matrix Y, such that $F = X+ZY$, then compute K_1 and K from (14) and (17).

Step 3 Compute

$$J = -\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = K_1 + KF, \quad J_1 \in R^{(n-q)\times(q-p)}, \quad J_2 \in R^{p\times(q-p)}, \quad R_1 = (I_q \;\; 0)(R - \widetilde{B}K).$$

Taking

$$V = Q\begin{bmatrix} R_1 \\ J_1 \end{bmatrix}, \quad W = J_2.$$

3.3 Regional pole placement in LMI regions

In this section, we will discusse the solution of matrix equation $AV + BW = EVF$, such that $\sigma(F) \subseteq \Theta = \{\lambda \in C \mid f_\Theta = \Delta + \lambda L + \overline{\lambda} L^T\}$, where $\Delta = \Delta^T \in R^{(q-p)\times(q-p)}$ and $L = L^T \in R^{(q-p)\times(q-p)}$.

Theorem 2 Assume that system (1) is Γ-strongly controllable, then there exist a solution to $AV + BW = EVF$ with $\sigma(F) \in \Theta$ if and only if there exist a symmetric positive matrix $M \in R^{(q-p)\times(q-p)}$ and a matrix $N \in R^{(q-p)\times(q-p)}$ such that

$$\Delta \otimes M + L \otimes (MX^T) + L \otimes (NZ^T) + L^T \otimes (MX^T)^T + L^T \otimes (NZ^T)^T < 0, \qquad (23)$$

where

$$X = M_1 A_C R - M_1 A_C \widetilde{B}(T\widetilde{B})^+ TR, \quad Z = M_1 = M_1 A_C \widetilde{B}(I - (T\widetilde{B})^+(T\widetilde{B})),$$

and $\otimes$ is the Kronecker's product. In this case $F = X + ZY$ with $Y = N^T M^{-T}$.

Proof It holds that $\sigma(F) \subseteq \Theta$ if and only if $\sigma(F^T) \subseteq \Theta$. Since $\sigma(F^T) \subseteq \Theta$, there exists a symmetric positive definite matrix $M \in R^{(q-p)\times(q-p)}$ such that (see [8])

$$\Delta \otimes M + L \otimes (MF^T) + L^T(MF^T)^T < 0, \tag{24}$$

From the above result, we have, and by substituting this value into (25), and by putting $N = MY^T$, we obtain the desired result.

4 An illustrative example

Consider the matrices E, A, B and C defined below, corresponding to a Γ-strongly-controllable, whose finite poles are given by $\sigma(E, A) = \{j, j\}$.

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Let $P = Q = I$, then $E_C = E$, $A_C = A$, $T = (0 \ \ 0 \ \ 0 \ 1)$. From (8) and (9), we obtain

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

For any given $F \subset \Gamma$, for example, let

$$F = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

using the pole placement method in context state space systems to compute the gain matrix

$$Y = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix},$$

such that $F = X + ZY$, we obtain that

$$K = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Compute

$$J = -\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = K_1 + KF = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}, \quad R_1 = (I_q \ \ 0)(R - \widetilde{B}K) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$V = Q\begin{bmatrix} R_1 \\ J_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad W = J_2 = (-1 \ \ 1).$$

Now it is easy to illustrate that, for $F = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, V, W is the solution to the matrix equation $AV + BW = EVF$.

5 Conclusion

In this paper, the solution of the matrix equation $AV + BW = EVF$ associated to singular systems has been settled by a new method. A numerical example has shown that the new method can provide an interesting framework to implicitly and optimized solutions for the considered constrained Sylvester equation. This solution of this problem can also be obtained by regional pole placement in LMI regions.

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关于广义系统中的Sylevster方程 $AV + BW = EVF$ 的解

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摘　要: 本文考虑了在广义系统中的Sylevster方程 $AV + BW = EVF$ 的解的问题，这一矩阵方程与相关矩阵的秩及区域极点配置有关。在广义系统是Γ-强能控制的假设下，通过一系列的辅助变换，我们把所考虑的Sylevster方程的解的问题转化为解一个可控制的降阶标准系统的Sylevster方程来解决。此外，所考虑的Sylevster方程的解也可通过用复域的LMI区域的极点配置的方法来给出。本文的结果对计算和分析这类方程的解带来很大的方便。

关键词: 广义系统；Sylevster方程；区域极点配置